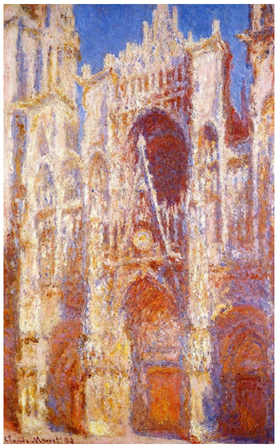

Catedral de Rouen
Claude Monet

Nota. *Estimados lectores hace algunos meses tuvimos un breve pero intenso e interesante intercambio de correos con dos queridas colegas, la profesora Gaby Campero y la profesora Pilar Valencia. El meollo de la discusión fue la equivalencia entre el principio de inducción y el principio del buen orden.*
No parece aventurado decir que la mayoría de los profes y ayudantes del Departamento de Matemáticas, y la mayoría de nuestros estudiantes, tenemos en nuestra memoria inmediata una lista de herramientas muy útiles a la hora de enfrentarnos a los desafíos del día a día. En esa lista está, sin duda, la equivalencia de estos dos principios.
Bueno, pues resulta que no siempre son equivalentes. Que la posibilidad de demostrar la equivalencia descansa fuertemente en los axiomas previos con los que hayamos definido a los números naturales.
Ya en el libro Curso introductorio de álgebra, Tomo I, *de Diana Avella y Gaby Campero, las autoras nos habían advertido que en algunos libros aparecen "demostraciones" erróneas de que el principio del buen orden implica el principio de inducción, afirmando así que ambos son equivalentes. En el contexto de los axiomas de Peano, esto no es cierto.*
¡Guau! Nuestra sorpresa fue inmensa. Claro, una vez que nos repusimos de la sacudida nació una urgente necesidad de saber más, de comprender, a pesar de que no somos algebristas, en dónde estaba esa sutileza por la cual pasamos sin darnos cuenta.
Gaby Campero nos sugirió darle una leída al escrito que a continuación reproducimos.

Are Induction and Well-Ordering Equivalent?.
Öhman, Lars–Daniel (6 May 2019).
The Mathematical Intelligencer.
41(3): 33–40.

Ojalá disfruten este texto tanto como nosotros.

Are Induction and Well-Ordering Equivalent? I

Lars–Daniel Öhman

Over the course of the last few years, I have used in my teaching a variety of textbooks in the very broadly constructed area of discrete mathematics. In one of these, the delightful *Discrete Mathematics*, by Norman L. Biggs, second edition, which I used in an introductory course in discrete mathematics, there is a section on how to introduce the natural numbers axiomatically, and in one subsection, the principle of mathematical induction is introduced as an axiom. This axiom is then used in a subsection on greatest and least members to prove that every nonempty subset of the natural numbers has a least member. The fact that the natural numbers have this property is usually called the well-ordering principle. It should be noted that sometimes this term is used for the well-ordering theorem, which states that every set can be well-ordered.

It seemed to me that there was something missing from the exposition, namely a remark along the lines that "we could have taken the well-ordering principle as an axiom instead of induction and still gotten the same structure, namely the familiar natural numbers." True enough, when I went back to the first (revised) edition of Biggs's book, which I had read for the introductory discrete mathematics course I took as an undergraduate, I found that the well-ordering principle was taken as an axiom, and the induction principle was a theorem. This would seem to indicate that either way of presenting things would yield the same outcome.

At this point, I decided to try to find a more solid, explicit, and detailed source for the equivalence of the induction principle and the well-ordering principle. So I searched for relevant articles in the MathSciNet database, and found the seemingly aptly titled article *The equivalence of the multiplication, pigeonhole, induction, and well-ordering principles* and the related article *Placing the pigeonhole principle within the defining axioms of the integers*. Unexpectedly, what piqued my interest was not the articles themselves, but rather the reviews by Perry Smith. According to the Mathematics Genealogy Project database, Smith is an academic grandson of the famous logician Stephen Kleene (whose doctoral dissertation, incidentally, is titled *A Theory of Positive Integers in Formal Logic*), completing his PhD at UCLA in 1970 with a thesis entitled *Some Contributions to Montague's Abstract Recursion Theory*. Smith's publication record in MathSciNet comprises only two papers, but he has reviewed at least 169 papers in the database.

From the review of the article mentioned first, what I took away was the remark that "The authors work in Zermelo–Fraenkel set theory, but such arguments should be given in a weaker system of set theory or arithmetic in which the principles in question are not theorems". The other review deserves to be cited in full:

"The authors argue very informally that the pigeonhole principle can replace the induction axiom or the well-ordering principle in the set-theoretic characterization of the natural numbers. However, this claim must be formulated carefully if it is to be correct. For example,
(1) the ordinals less than $\omega + \omega$ satisfy the first four (Dedekind–) Peano postulates and the well-ordering principle, but not the induction axiom or the pigeonhole principle;
(2) the class of all cardinals satisfies the first four Peano postulates, the well-ordering principle, and the pigeonhole principle (if m elements are distributed into n boxes and $m > n$ then two elements must go into the same box), but not the induction axiom."

I realized then that I had long suffered from a fundamental misconception. In the remainder of the present article, I will expand on the issue of what exactly

my misconception consisted of, indicate that I was not alone in this misconception, and call on all good forces to help work against the further spread of it.
Regarding rigor, I do not aim to give a logically watertight presentation of the question at hand, but rather to supply a conceptual exposition of the ideas involved with enough details that trained mathematicians should feel able to bridge the technical gaps on their own, and that the interested amateur will at least be able to grasp what is at stake in these questions. In particular, I will not go into the question? of first-order versus second-order versions of the induction axiom.

A Fun Game to Illustrate What Is Going On
Here's a game, the relevance of which will soon become clear, which I used to play with my three-year-old daughter. I would tell her a set of properties of a thing in the house, and she would guess which thing I was thinking of. As an example of an instance of the game, I would tell her that I'm thinking about something that

• has four legs,
• has a long tail,
• sleeps in her bed,
• is dark brown.

What am I thinking about? The answer, as she would tell me after a moment's thought, was her plush toy "Mousey." In mathematical terminology, we say that Mousey is a model for the set of clues, that is, a concrete example of something that satisfies all the properties. In fact, these four clues singled out a unique object -unique up to isomorphism, that is. Mousey was bought at Ikea, and has in fact been lost and replaced by an isomorphic copy on at least one occasion, unbeknownst to my daughter. In mathematical terminology, we say that these properties are categorical, that is, that they admit only one model.
It would then be her turn to give the clues, and often they would go something like this: I'm thinking about ...

• Daddy,
• who is sitting at the table,
• having breakfast.

The right answer, as I would figure out after a moment's thought, was indeed "Daddy."
In the following rounds of the game, I would vary the clues slightly, sometimes again singling out Mousey uniquely, and sometimes singling out uniquely another plush toy, "Tigris" (who, of course, is a tiger).
For instance, if I retained the first three clues, then the clue "is dark brown" could be supplanted with "has rodent teeth," and the set of clues would still single out Mousey. In mathematical terminology, we say that the clue "has rodent teeth" is equivalent to the clue "is dark brown" relative to the first three clues.
As another example, replacing "is dark brown" with "has sharp teeth" renders the set of clues no longer categorical, since both Tigris and Mousey are compatible with all of the clues. Therefore, the clues "is dark brown" and "has sharp teeth" are not equivalent relative to the first three clues. In mathematical terminology, we say that the set of clues has nonisomorphic models, the models being Mousey and Tigris.
On my daughter's turn to give clues, since "Daddy" usually remained the first clue, the question of which of the other clues were equivalent to each other given some base set of clues was less meaningful, since any true propositions about me would in some vacuous sense be equivalent to each other, they could replace each other freely, without affecting the categoricity of the set of properties. The only possible model would still be Daddy.

Continuará

Referencias:
Norman Linstead Biggs. *Discrete Mathematics*, revised ed. Oxford Science Publications, Oxford University Press, 1989.
Norman Linstead Biggs. *Discrete Mathematics*, second ed. Oxford Science Publications, Oxford University Press, 2002.
Rodney T. Hansen and Leonard George Swanson. *Placing the pigeonhole principle within the defining axioms of the integers*. In Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979), Congress. Numer., XXVI, Utilitas Math., Winnipeg, Man., 1980, pp. 183–186.